Homogeneous DE

An equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ in which f(x,y) and g(x,y) are homogeneous functions of x and y of the same degree is called homogeneous first order differential equation.

Examples:

(i) $\frac{dy}{dx} = \frac{y^3+3x^2y}{x^3+3xy^2}$

(ii) $(x^2 + y^2)dy = xydx$

Structure:

We can solve it using Separation of Variables by putting $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ [use **uv** rule]

Now put the value of $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the given DE then using variable separation we get the solution.

Solve: $\mathbf{\frac{dy}{dx} = \frac{x^2+y^2}{xy}}$

Taking $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now put the value of $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the given DE we get

$v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{x.vx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{v}$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1}{v}$

$\Rightarrow vdv = \frac{1}{x}dx$ [variable separation]

Then integrate this we get

$$\int vdv = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{v^2}{2} = \ln x + c$$

$$\Rightarrow v^2 = 2(\ln x + c)$$

$$\Rightarrow v^2 = 2(\ln x + \ln c) \quad \text{[taking c=lnc]}$$

$$\Rightarrow v^2 = 2(\ln cx)$$

$$\Rightarrow v = \sqrt{2(\ln cx)}$$

$$\Rightarrow \frac{y}{x} = \sqrt{2(\ln cx)}$$

$$\Rightarrow y = x\sqrt{2(\ln cx)}$$

Solve: $\mathbf{\frac{dy}{dx} = \frac{y(x-y)}{x^2}}$

Taking $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now put the value of $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in the given DE we get

$v + x\frac{dv}{dx} = \frac{vx(x-vx)}{x^2}$

$\Rightarrow v + x\frac{dv}{dx} = v(1 - v)$

$\Rightarrow x\frac{dv}{dx} = v(1 - v) - v$

$\Rightarrow x\frac{dv}{dx} = -v^2$

$\Rightarrow -\frac{dv}{v^2} = \frac{1}{x}dx$ [variable separation]

Then integrate this we get

$$\int -\frac{dv}{v^2} = \int \frac{1}{x}dx$$

$$\Rightarrow \frac{1}{v} = \ln x + c$$

$$\Rightarrow v = \frac{1}{\ln x + c}$$

$$\Rightarrow v = \frac{1}{\ln x + \ln c}$$

[taking c=lnc]

$$\Rightarrow v = \frac{1}{(\ln cx)}$$

$$\Rightarrow \frac{y}{x} = \frac{1}{(\ln cx)}$$

$$\Rightarrow y = \frac{x}{(\ln cx)}$$

Exercise :	**Answer**
(i) $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$	**(i)** $x = c(x^2 - y^2)$
(ii) $(x^2 + y^2)dx + 2xydy = 0$	**(ii)** $x\left(1 + \frac{3y^2}{x^2}\right)^{\frac{1}{3}} = c$
(iii) $(x^2 + y^2)dy = xydx$	**(iii)** $x^2 = 2y^2\,(\ln y + c)$
(iv) $(x^2 - y^2)\frac{dy}{dx} = xy$	**(iv)** $\ln\frac{y}{x} + \frac{1}{2}\,\frac{x^2}{y^2} = \ln x + c$